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Solvability Theorems Involving Inf-Convex Functions

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This paper presents a unified theory on solvability of certain general systems of inequalities involving functions expressible as the pointwise infimum of convex functions. The approach used to develop these solvability theorems relies on Minkowski duality. Extensions of Farkas' lemma and other solvability theorems are developed, both with and without a regularity condition, with applications to optimization. © 1995 Academic Press, Inc.

1. INTRODUCTION

Solvability theorems deal with verifiable conditions, often in terms of subdifferentials, which ensure that a system of equations or inequations is consistent. One significant area of application of such theorems is in optimization theory, particularly in relation to the derivation of necessary conditions for optimality (see [3, 9, 22, 26]). The object of this paper is to present a range of solvability theorems for general systems of nonlinear

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inequalities. The functions involved will be assumed to be expressible as the pointwise infimum of a family of convex functions.

Several authors have discussed duality frameworks for general classes of functions expressible as the pointwise infimum of a family of functions; the seminal work in this area is Kutateladze and Rubinov [19] (see also [18] and [20]) with regard to *Minkowski duality*. Recall that for a function $f: X \to \mathbb{R}$ and a class $\mathcal{H}$ of functions we define the set of *$\mathcal{H}$-minorants* of f as the set

$$\mathcal{H}f = \{h \in \mathcal{H} : (\forall x \in X)\, h(x) \leq g(x)\}.$$

This set $\mathcal{H}f$ was called the *support set* of f in [19]. Minkowski duality concerns the mapping $f \mapsto \mathcal{H}f$ and conditions under which such a set-valued mapping is nonempty valued. Clearly the set $\mathcal{H}f$ is a "global" concept (whereas the usual subdifferential of a convex function is a "local" concept) and we shall use such sets to derive dual conditions characterizing solvability of certain nonlinear systems. We will be particularly interested in the case in which the class $\mathcal{H}$ consists of *affine* or, more generally, *convex functions*. Thus the object of this paper is to use the Minkowski duality framework to discuss solvability of systems of *inf-convex* functions. We present these conditions in terms of sets of *affine minorants*. The principle results are established first in an asymptotic form without a regularity condition and then more precise results are obtained using an appropriate *closure regularity condition*. We obtain extensions to the classical solvability result known as Farkas' lemma which unifies a number of recent extensions; see [8–11, 27] and others. Nonconvex versions of Farkas' lemma have been used recently by several authors to obtain necessary optimality conditions for quasi-differentiable programming problems (see [9, 11]). In addition, the most recent nonhomogeneous versions have been applied to a large class of global optimization problems (see [14]).

The results of this paper will be applied in subsequent work to specific problems in infinite-dimensional optimization and functional analysis. Here we briefly consider an application to Lagrangian duality extending the work of [15].

In Section 2 we present the basic definitions and provide fundamental results on classes of inf-convex functions utilizing the work of [19]. In addition we develop a general extension of Farkas' lemma. In Section 3 we establish solvability results for systems of inf-convex functions without recourse to a regularity condition. In Section 4 we derive related results assuming a regularity condition suggested in [15]. Finally, in Section 5 we apply the results of Section 4 to Lagrangian duality for certain specially structured nonconvex programming problems.

2. Preliminary Results

Throughout this section X_0 will denote a nonempty convex subset of X, where X is a locally convex Hausdorff topological vector space (l.c.t.v.s). Let $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ and let $\Gamma(X_0)$ denote the set of all lower semicontinuous (l.s.c) convex functions $h: X_0 \to \mathbb{R}_{+\infty}$ with dom $h = \{x \in X_0 : h(x) < +\infty\}$ (the *effective domain* of h).

Definition 2.1. A function $f: X_0 \to \mathbb{R}$ is said to be *inf-convex*, if there exists a nonempty index set Δ and a family of functions $(f_\alpha)_{\alpha\in\Delta}$ with $f_\alpha \in \Gamma(X_0)$, such that, for each $x \in X_0$,

$$f(x) = \inf_{\alpha\in\Delta} f_\alpha(x). \tag{1}$$

Similarly, we will say that f is *min-convex* if "inf" is replaced by "min" in (1). The function f will be called the *lower envelope* of the family of functions f_α. If f is inf-convex it possesses an *exhaustive* family of upper convex approximations [5].

These concepts have been used previously in the study of certain problems in functional analysis by Kutateladze and Rubinov [20]. In the terminology of [20] let $\mathcal{H}$ be a collection of real-valued functions defined on X_0. Then a real-valued function f defined on X_0 is *$\mathcal{H}$-convex* if there exists a set $\mathcal{U} \subseteq \mathcal{H}$ such that, for all $x \in X_0$,

$$f(x) = \sup_{h\in\mathcal{U}} h(x).$$

The class of *$\mathcal{H}$-concave* functions is defined symmetrically (with sup replaced by inf). Clearly, if $\mathcal{H}$ is the class of all convex functions on X_0 then $\mathcal{H}$-concavity and inf-convexity coincide.

We begin with several examples to illustrate the broad nature of the class of inf-convex and min-convex functions.

Example 2.1. (i) Let f: $\mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{|x|}$; then f is inf-convex and not min-convex since

$$f(x) = \inf_{a>0} \left[\frac{|x|}{2\sqrt{a}} + \frac{\sqrt{a}}{2} \right].$$

Note that $f_a(x) = |x|/2\sqrt{a} + \sqrt{a}/2$ and $f_a(0) > f(0)$ for all $a \in \Delta = (0, +\infty)$. Clearly f is not convex nor locally Lipschitz.

(ii) Every function $f \in \Gamma(X_0)$ is min-convex with the index set Δ a singleton.

(iii) Every concave function is the infimum of its affine majorants and is hence inf-convex.

(iv) Let f be Lipschitz on X_0; then f is min-convex. This follows since for all x, $y \in X_0$ and some constant $K > 0$ we have

$$|f(x) - f(y)| \leq K\|x - y\|.$$

Thus, for each $y \in X_0$, the function $x \mapsto f(y) + K\|x - y\|$ is convex and

$$f(x) = \min_{y \in X_0} (f(y) + K\|x - y\|).$$

(iv) Every piecewise linear function is min-convex.

(v) The class of min-convex functions is closed under the operations of pointwise minima and positive linear combination.

(vi) Let $f = f_1 - f_2$ where f_1 is convex and f_2 is a convex function defined as the pointwise maximum of a family of affine functions; that is, for each $x \in X_0$,

$$f_2(x) = \max_{\alpha \in \Delta} f_{2,\alpha}(x),$$

where $f_{2,\alpha}$ is affine for each $\alpha \in \Delta$. Thus f is a difference convex function and

$$f(x) = \min_{\alpha \in \Delta} (f_1(x) - f_{2,\alpha}(x)).$$

Thus, since $f_1 - f_{2,\alpha}$ is convex for each α, f is min-convex.

(vii) An important special case of (vi) above occurs when f_2 is a continuous sublinear function, in this case $f_2(x) = \max_{v \in \partial f_2(0)} v(x)$ (where $\partial f_2(0)$ denotes the subdifferential of f_2 at 0). Hence

$$f(x) = \min_{v \in \partial f_2(0)} (f_1(x) - v(x)).$$

Thus f is min-convex. If f_2 is merely lower semicontinuous rather than continuous then f can be shown to be inf-convex. In this case f is almost-DC in the sense of [10].

(viii) Let f be a positively homogeneous (of degree one), continuous, and quasi-convex function. Then Crouzeix [4] has shown that f is min-convex with the index set Δ containing only two elements. In particular, if $h : X \to \mathbb{R}$ is quasi-convex and directionally differentiable at $a \in X$, then $h'(a, \cdot)$ is min-convex (and quasi-convex) but in general not convex (see [4] for details).

(ix) Consider the parametrized optimization problem, for $x \in \mathbb{R}^n$,

$$(\mathrm{P}_x) \qquad \psi(x) = \max\{f(x, y) : g_i(x, y) \leq 0, i \in I, h_j(x, y) = 0, j \in J\},$$

where $I = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, q\}$, and $y \in \mathbb{R}^m$. As in Ishizuka [11] we denote the feasible region of (P_x) by $R(x)$, $Y(x) = \{y \in R(x) : f(x, y) = \psi(x)\}$, and

$$K(x, y) = \{\mu \in \mathbb{R}^{p+q} : \nabla_y L(x, y, \mu) = 0, \mu_i g_i(x, y) = 0, \ \mu_i \geq 0, i \in I\},$$

where the Lagrangian L is defined as

$$L(x, y) = f(x, y) + \sum_{i=1}^{p} \mu_i g_i(x, y) + \sum_{j=p+1}^{p+q} \mu_j h_j(x, y).$$

Now under suitable differentiability assumptions on the objective and constraint functions and assuming both the constant rank and Mangasarian–Fromowitz constraint qualifications, it follows (see [7, 13]) that ψ is Lipschitz near $\bar{x}$ and directionally differentiable at $\bar{x}$, $K(\bar{x}, Y(\bar{x}))$ is compact, and

$$\psi'(\bar{x}; d) = \min_{\bar{y} \in Y(\bar{x})} \max_{\mu \in K(\bar{x}, \bar{y})} \nabla_y L(\bar{x}, \bar{y}, \mu)(d).$$

In particular, the value function ψ has a directional derivative which is min-convex. This observation has been used in [11] to develop optimality conditions for min-max problems.

(x) In Borwein [2] various duality results were established for the infimum of a family of convex programs. In particular, Borwein considers optimization problems of the form

$$(\mathrm{P}_i) \qquad \inf_{x_i} F_i(x_i, y_i) = h_i(y_i), \qquad y_i \in Y_i,$$

where $i \in I$, an arbitrary index set; X_i and Y_i are locally convex Hausdorff topological vector spaces; and $F_i : X_i \times Y_i \to [-\infty, \infty]$. It is not difficult to show that if F_i is convex (on the product space $X_i \times Y_i$) then the function h_i is convex.

We will now give a description of the class of inf-convex functions. The following lemma will be useful in the following: see [19, Theorem 2.2] for the proof. For convenience denote the set of all neighborhoods of a point $x \in X$ by $\mathcal{N}(x)$.

LEMMA 2.1. *Let H be a convex cone of upper semicontinuous (u.s.c) functions such that the following is satisfied:*

$$\begin{gathered}(\forall \varepsilon > 0)(\forall x_0 \in X_0)(\forall V \in \mathcal{N}(x_0))(\exists h \in H) \\ h(x_0) < -(1-\varepsilon),\ h(x) \geq -1 \text{ (for all } x \in V),\ h(x) \geq 0 \text{ (for all } x \notin V).\end{gathered} \tag{2}$$

If a function f defined on X_0 is u.s.c and bounded from above by a function $h \in H$ then f has the following form for all $x \in X_0$:

$$f(x) = \inf\{h(x) : h \in H, h \geq f\}.$$

THEOREM 2.1 *Let $f : X_0 \to \mathbb{R}$ be an u.s.c function such that, for some continuous function $h \in \Gamma(X_0)$, $h(x) > f(x)$ for all $x \in X_0$. Then f is inf-convex on X_0.*

Proof. Let $\mathcal{H} = \{h \in \Gamma(X_0) : h \text{ is continuous on } X_0\}$. It suffices to show that (2) holds and the result will follow directly by Lemma 2.1. Let $\varepsilon > 0$, $x_0 \in X_0$, and $V \in \mathcal{N}(x_0)$. Then $V = x_0 + U$ for some neighbourhood U of zero. Also, there is a semi-norm p such that $U = \{u \in X : p(u) \leq 1\}$. We will assume, without loss of generality, that U is a closed symmetric neighbourhood of zero for convenience. Let $h(x) = -1 + p(x - x_0)$. Clearly $h \in \mathcal{H}$ and $h(x_0) = -1 < -(1-\varepsilon)$. Since $p(x - x_0) \geq 0$ and $p(x - x_0) > 1$ for $x \notin V$ we have $h(x) \geq -1$ if $x \in V$ and $h(x) > 0$ if $x \notin V$. ∎

Remark 2.1. Since the lower envelope of a family of continuous functions is clearly upper semicontinuous, the following are equivalent for a function f:

(i) there is a family $(h_\alpha)_{\alpha \in \Delta}$ with $h_\alpha \in \Gamma(X_0)$ and continuous on X_0, for each $\alpha \in \Delta$, such that $f(x) = \inf_{\alpha \in \Delta} h_\alpha(x)$, for all $x \in X$;

(ii) f is upper semicontinuous and there is a function $h \in \Gamma(X_0)$, continuous on X_0, such that $f < h$.

Using Lemma 2.1 it is not difficult to give examples of *thin* sets $\mathcal{H} \subseteq \Gamma(X_0)$ (for example, sets which can be described using very few parameters) with the property that each function f satisfying (ii) above can be represented in the form

$$f(x) = \inf\{h(x) : h \in \mathcal{H}, h \geq f\}.$$

This problem is discussed in detail in [19] (see also Balder [1]).

It is considerably more difficult to describe the min-convex functions. In Lemma 2.2 below we provide a description in the special case of min-sublinear functions.

We now provide a number of important definitions to be used in the next section. For the l.c.t.v.s. X we denote the *conjugate* (dual) space by X'. Recall that the *subdifferential* of a convex function $f: X \to \mathbb{R}_{+\infty}$ at $x_0 \in X$ is the set

$$\partial f(x_0) = \{v \in X' : (\forall x \in X) f(x) - f(x_0) \geq v(x - x_0)\}.$$

The subdifferential of a convex function is a local concept. In the following we define related concepts using sets of affine and convex (rather than linear) minorants and majorants to obtain a "global" approximating tool for convex and concave functions.

DEFINITION 2.2. Let $p : X \to \mathbb{R}_{+\infty}$ be a l.s.c. sublinear function. Then the *subdifferential* of p is the set

$$\underline{\partial} p = \{v \in X' : (\forall x \in X)\, v(x) \leq p(x)\}.$$

clearly $\underline{\partial} p$ coincides with usual subdifferential $\partial p(0)$. It is well known that $\underline{\partial} p \neq \emptyset$ and, for all $x \in X$, $p(x) = \sup_{v \in \underline{\partial} p} v(x)$ (that is, p is the support function of its subdifferential). Furthermore, define, for each $x_0 \in X$,

$$\underline{\partial} p(x_0) = \{v \in \underline{\partial} p : v(x_0) = p(x_0)\}.$$

It is straightforward to show that $\underline{\partial} p(x_0) = \partial p(x_0)$. This notation for the subdifferential of a sublinear function as the set of linear minorants and the distinction between the global subdifferential and its local pointwise condition is used extensively in [5] (see also [6, 21]).

In the following we identify the set of all continuous *affine* functions defined on X with the set $X' \times \mathbb{R}$.

DEFINITION 2.3. Let $f: X \to \mathbb{R}_{+\infty}$ be a l.s.c. convex function. Then the set of *affine minorants* of f is the set

$$\mathcal{A} f = \{(v, c) \in X' \times \mathbb{R} : (\forall x \in X)\, v(x) + c \leq f(x)\}.$$

It is well known that, for each $x \in X$, $f(x) = \sup_{(v,c)\in\mathscr{A}f}(v(x) + c)$. The set of affine minorants which coincide with f at a point $x_0 \in X$ is defined as follows:

$$\mathscr{A}f(x_0) = \{(v, c) \in \mathscr{A}f: v(x_0) + c = f(x_0)\}.$$

It is easy to check that

$$\mathscr{A}f(x_0) = \{(v, c) \in X' \times \mathbb{R} : v \in \partial f(x_0), c = f(x_0) - v(x_0)\}$$

where $\partial f(x_0)$ denotes the subdifferential of f at x_0.

Symmetrically we can consider the set of *affine majorants* of an u.s.c. concave function g as the set

$$\overline{\mathscr{A}}g = \{(v, c) \in X' \times \mathbb{R} : (\forall x \in X)\, v(x) + c \geq g(x)\}.$$

By analogy with the above we shall define the set of *convex majorants* $\overline{\mathscr{C}}f$ of an inf-convex function f as follows:

$$\overline{\mathscr{C}}f = \{h \in \Gamma(X_0) : (\forall x \in X_0)\, h(x) \geq f(x)\}.$$

Similarly, define the set of convex majorants which coincide with f at $x_0 \in X_0$ as

$$\overline{\mathscr{C}}f(x_0) = \{h \in \overline{\mathscr{C}}f : h(x_0) = f(x_0)\}.$$

Clearly f is min-convex if and only if $\overline{\mathscr{C}}f(x)$ is non-empty for all $x \in X_0$.

The study of l.s.c. convex functions can be substantially simplified by considering certain related sublinear functions defined on appropriate cones. Recall the following well known construction of a l.s.c. sublinear function from a l.s.c. convex function $h : X \to \mathbb{R}_{+\infty}$. Define the sublinear function $\tilde{H}$ on $X \times \mathbb{R}$ by

$$\tilde{H}(x, \gamma) = \begin{cases} \gamma h(x/\gamma) & \text{if } \gamma > 0 \\ +\infty & \text{if } \gamma \leq 0 \end{cases}$$

Let $H = \operatorname{cl} \tilde{H}$ (see Rockafellar [25] for a discussion of closure of convex functions). It is clear that H is a sublinear function and, since h is l.s.c., we have $H(x, \gamma) = \tilde{H}(x, \gamma) = \gamma h(x/\gamma)$ if $\gamma > 0$. Similarly, we can construct a convex cone from a convex set as follows. Let $K = \operatorname{cone}(\operatorname{dom} h \times \{1\})\backslash\{(0, 0)\} = \{(\mu x, \mu) : x \in \operatorname{dom} h, \mu > 0\}$. Clearly $K = \operatorname{dom} \tilde{H} \subseteq \operatorname{dom} H \subseteq \operatorname{cl} K$. It follows easily that $\underline{\partial} H = \mathscr{A}h$ where $\mathscr{A}h$ denotes the set of affine minorants of h (so that the subdifferential of H coincides with the set of affine minorants of h).

This method can be extended to the study of inf-convex and min-convex functions.

DEFINITION 2.4. Let K be a *conic set* (i.e., $\gamma K \subseteq K$ for all $\gamma > 0$). Then subl(K) denotes the set of all l.s.c. sublinear functions $p: X \to \mathbb{R}_{+\infty}$ with $K \subseteq \operatorname{dom} p$. A function f defined on a conic set $K \subseteq X$ will be called inf-SL (min-SL) if there is a family $(p_\alpha)_{\alpha\in\Delta}$, where $p_\alpha \in \text{subl}(K)$, such that $f(x) = \inf_{\alpha\in\Delta} p_\alpha(x)$ $(f(x) = \min_{\alpha\in\Delta} p_\alpha(x))$ for all $x \in K$.

Let f be an inf-convex function with, for each $x \in X_0$, $f(x) = \inf_{\alpha\in\Delta} h_\alpha(x)$ and $h_\alpha \in \Gamma(X_0)$. Let $K = \{(\mu x, \mu): x \in X_0, \mu > 0\}$ and define $F(x, \gamma)$ on K by $F(x, \gamma) = \gamma f(x/\gamma)$ for $(x, \gamma) \in K$. Similarly, let

$$\tilde{H}_\alpha(x, \gamma) = \begin{cases} \gamma h_\alpha(x/\gamma) & \text{if } \gamma > 0 \\ +\infty & \text{if } \gamma \leq 0 \end{cases}$$

and $H_\alpha = \operatorname{cl} \tilde{H}_\alpha$. Since $h_\alpha \in \Gamma(X_0)$ it follows that $H_\alpha \in \text{subl}(K)$. In addition, we have

$$F(x, \gamma) = \inf_{\alpha\in\Delta} H_\alpha(x, \gamma).$$

This discussion shows that a study of inf-convex functions defined on X_0 can be carried out using inf-SL functions defined on the cone K. A similar analysis follows for min-convex and min-SL functions.

Note that $K \neq X \times \mathbb{R}$ even in the case in which $X_0 = X$. Hence the set subl(K) contains sublinear functions p with $\operatorname{dom} p \neq X \times \mathbb{R}$.

A continuous sublinear function p defined on X with $\operatorname{dom} p = X$ possesses the important property that the subdifferential ∂p is a weak* compact convex subset of X'.

DEFINITION 2.5. A function f defined on a conic set $K \subseteq X$ will be called inf-$\text{SL}_C(X)$ (min-$\text{SL}_C(X)$) if there is a family $(p_\alpha)_{\alpha\in\Delta}$, with p_α continuous and $p_\alpha \in \text{subl}(X)$, such that $f(x) = \inf_{\alpha\in\Delta} p_\alpha(x)$ $(f(x) = \min_{\alpha\in\Delta} p_\alpha(x))$ for all $x \in K$.

An important group of min-$\text{SL}_C(X)$ functions defined on X are the so-called difference sublinear (DSL) functions. A function f is DSL if $f = p_1 - p_2$ where, for each $i = 1, 2$, p_i is a continuous sublinear function defined on X. It has been shown by Dem'yanov and Rubinov [5], assuming X is a Hilbert space, that every continuous positively homogeneous (of degree one) function is inf-SL. The following lemma provides an interesting characterization of min-$\text{SL}_C(X)$ functions in a normed space.

LEMMA 2.2. *Let X be a normed space and let $\mathcal{P}$ denote the class of all positively homogeneous functions defined on X. Then*

$$\text{min-}SL_C(X) = \{f \in \mathcal{P} : (\forall a \in X)(\exists k > 0)(\forall x \in X) f(x) - f(a) \leq k\|x - a\|\}.$$

Proof. Let $f \in \text{min-SL}_C(X)$, $a \in X$, and suppose that p_α is a continuous upper sublinear bound to f such that $f(a) = p_\alpha(a)$. Thus,

$$f(x) - f(a) \leq p_\alpha(x) - p_\alpha(a) \leq \|p_\alpha\| \, \|x - a\|,$$

where $\|p_\alpha\| = \sup\{|p_\alpha(x)| : \|x\| \leq 1\}$ (this exists since p_α is continuous and sublinear and hence bounded above on the unit ball). Clearly $f \in \mathcal{P}$.

Conversely, let $f \in \mathcal{P}$ and $f(x) - f(a) \leq k\|x - a\|$ for all $x \in X$; we can, without loss of generality, assume $a \neq 0$. Define $g(x) = f(a) + k\|x - a\|$. Clearly g is a convex function with $f(x) \leq g(x)$ and $f(a) = g(a)$. Let $\|a\| = r > 0$ and take $x \in X \backslash \{0\}$, then

$$f(x) = \frac{\|x\|}{r} f\left(r \frac{x}{\|x\|}\right) \leq \frac{\|x\|}{r} g\left(r \frac{x}{\|x\|}\right).$$

Define $p(x) = (\|x\|/r) g(r(x/\|x\|))$. It is not difficult to show that p is a continuous sublinear function, $p(a) = f(a)$, and $p(x) \geq f(x)$ for all $x \in X$. Consequently f is min-SL. ▮

Remark 2.2. As an immediate corollary of this result it follows that every positively homogeneous Lipschitz function is min-SL. It is worth noting that not every min-SL function is DSL. This follows since every DSL function is locally Lipschitz and directionally differentiable everywhere. However, not all locally Lipschitz positively homogeneous functions are directionally differentiable at all points; thus DSL $\neq$ min-SL. In addition, the function $k : \mathbb{R}^2 \to \mathbb{R}$, $k(x, y) = \sqrt{|x^2 - y^2|}$ is a continuous positively homogeneous function which is inf-SL but is not min-SL.

3. Solvability Theorems without a Regularity Condition

In this section we establish various solvability theorems for systems of inf-convex and related functions without assuming a regularity condition.

The following is a generalization of Farkas' lemma which applies to systems of inf-SL functions. Let X and Y be l.c.t.v.s., $K \subseteq X$ and $S \subseteq Y$ be closed convex cones. Denote by $S^* = \{v \in Y' : (\forall y \in S)\ v(y) \geq 0\}$ the *conjugate* (or dual) cone to S. Let $\Lambda \subseteq S^*$.

Let $g : K \to Y$ be a mapping with λg inf-SL on K for each $\lambda \in \Lambda$. Thus, for each $\lambda \in \Lambda$, there is a family $\{p_{\alpha_\lambda} : \alpha_\lambda \in \Delta(\lambda)\}$, with $p_{\alpha_\lambda} \in \text{subl}(K)$, such that, for all $x \in K$,

$$\lambda g(x) = \inf_{\alpha_\lambda \in \Delta(\lambda)} p_{\alpha_\lambda}(x).$$

Let $f: K \to \mathbb{R}$ be inf-SL on K with index set Δ. Hence there are functions $p_\alpha \in \text{subl}(X)$ such that, for all $x \in K$, $f(x) = \inf_{\alpha \in \Delta} p_\alpha(x)$.

In the following we often require *selections* from an indexed family of nonempty sets. Let $(A(t))_{t\in T}$ be a family of nonempty sets; then a *selection function* a is a function of the form:

$$a = (a_t)_{t \in T} \in \prod_{t \in T} A(t).$$

Thus, in particular, $a_t \in A(t)$ for each $t \in T$. We will use the notation

$$(a_t) \in \prod_{t \in T} A(t)$$

to indicate a selection function.

THEOREM 3.1. *Let $\Lambda \subseteq S^*$ and f and g be as above. Consider the following statements:*

(i) $x \in K,\ g(x) \in -S \Rightarrow f(x) \geq 0$.

(ii) *For each $\alpha \in \Delta$ and each selection $(\alpha_\lambda) \in \Pi_{\lambda \in \Delta} \Delta(\lambda)$ we have*

$$0 \in \text{cl}\left(\underline{\partial} p_\alpha + \text{co} \bigcup_{\lambda \in \Lambda} \underline{\partial} p_{\alpha_\lambda} - K^*\right).$$

Then (ii) *implies* (i). *If the set Λ is such that* cl co $\Lambda = S^*$ *then* (i) *implies* (ii).

Before establishing this theorem we require the following minimax result.

LEMMA 3.1. *Consider a set Λ and a family of sets $(\Delta(\lambda))_{\lambda \in \Lambda}$. For convenience let $\mathcal{S}_f$ denote the set of all selection functions a defined on Λ with, for each $\lambda \in \Lambda$, $a(\lambda) = \alpha_\lambda \in \Delta(\lambda)$. Assume that for each $\lambda \in \Lambda$ there is a function $t_\lambda : \Delta(\lambda) \to \mathbb{R}$ such that*

$$\inf_{\alpha_\lambda \in \Delta(\lambda)} t_\lambda(\alpha_\lambda) > -\infty.$$

Then for the function $t : \mathcal{S}_f \times \Lambda \to \mathbb{R}$, $t(a, \lambda) = t_\lambda(a(\lambda))$ we have

$$\inf_{a \in \mathcal{S}_f} \sup_{\lambda \in \Lambda} t(a, \lambda) = \sup_{\lambda \in \Lambda} \inf_{a \in \mathcal{S}_f} t(a, \lambda).$$

Proof. Since $\mathcal{S}_f$ is the set of all selection functions we have, for all $\lambda \in \Lambda$,

$$\inf_{a \in \mathcal{S}_f} t(a, \lambda) = \inf_{a \in \mathcal{S}_f} t_\lambda(a(\lambda)) = \inf_{\alpha_\lambda \in \Delta(\lambda)} t_\lambda(\alpha_\lambda).$$

Let $\varepsilon > 0$ and, for each $\lambda \in \Lambda$, let $\overline{\alpha}_\lambda \in \Delta(\lambda)$ be such that

$$t_\lambda(\overline{\alpha}_\lambda) \leq \inf_{\alpha_\lambda \in \Delta(\lambda)} t_\lambda(\alpha_\lambda) + \varepsilon.$$

Define the selection function $\overline{a} \in \mathcal{S}_f$ by $\overline{a}(\lambda) = \overline{\alpha}_\lambda$ for all $\lambda \in \Lambda$. Then

$$\begin{aligned}
\sup_{\lambda \in \Lambda} t(\overline{a}, \lambda) &= \sup_{\lambda \in \Lambda} t_\lambda(\overline{\alpha}_\lambda) \\
&\leq \sup_{\lambda \in \Lambda} \left(\inf_{\alpha_\lambda \in \Delta(\lambda)} t_\lambda(\alpha_\lambda) + \varepsilon \right) \\
&= \sup_{\lambda \in \Lambda} \inf_{a \in \mathcal{S}_f} t(a, \lambda) + \varepsilon.
\end{aligned}$$

Consequently,

$$\inf_{a \in \mathcal{S}_f} \sup_{\lambda \in \Lambda} t(a, \lambda) \leq \sup_{\lambda \in \Lambda} t(\overline{a}, \lambda) \leq \sup_{\lambda \in \Lambda} \inf_{a \in \mathcal{S}_f} t(a, \lambda) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary it follows that inf sup $\leq$ sup inf. The reverse inequality is always true and so the result follows. ∎

We now establish Theorem 3.1.

Proof. We begin by rewriting statement (ii) in an equivalent form. Let $\alpha \in \Delta$ and $a = (\alpha_\lambda) \in \prod_{\lambda \in \Lambda} \Delta(\lambda)$. Again the set of all selection functions will be denoted by $\mathcal{S}_f$ for convenience. Consider the sublinear function

$$p_{\alpha,a}(x) = \begin{cases} p_\alpha(x) + \sup_{\lambda \in \Lambda} p_{a(\lambda)}(x) & x \in K \\ +\infty & x \notin K. \end{cases}$$

Thus, $p_{\alpha,a} = p_\alpha + \sup_{\lambda \in \Lambda} p_{a(\lambda)} + \delta_K$, where δ_K is the indicator function of K. Since K is a closed convex cone, δ_K is a l.s.c. sublinear function. Hence $p_{\alpha,a}$ is a l.s.c. sublinear function.

Using subdifferential calculus we have (in particular since $\underline{\partial}\delta_K = -K^*$)

$$\underline{\partial} p_{\alpha,a} = \mathrm{cl}\left(\underline{\partial} p_\alpha + \mathrm{cl}\,\mathrm{co} \bigcup_{\lambda \in \Lambda} \underline{\partial} p_{\alpha(\lambda)} - K^* \right).$$

We are using here the well-known formulae that for l.s.c. sublinear functions p_1 and p_2, we have $\underline{\partial}(p_1 + p_2) = \text{cl}\ (\underline{\partial} p_1 + \underline{\partial} p_2)$ and $\underline{\partial}(\sup_\alpha p_\alpha) = \text{cl co} \bigcup_\alpha \underline{\partial} p_\alpha$. Both can be verified by a standard separation argument. Since, in general, cl $(C + \text{cl}\ D) = \text{cl}\ (C + D)$, we have

$$\underline{\partial} p_{\alpha,a} = \text{cl}\left(\underline{\partial} p_\alpha + \text{co} \bigcup_{\lambda\in\Lambda} \underline{\partial} p_{a(\lambda)} - K^*\right).$$

It is easily verified, by a separation argument, that $0 \in \underline{\partial} p_{\alpha,a}$ is equivalent to $p_{\alpha,a}(x) \geq 0$ for all $x \in K$. Therefore condition (ii) is equivalent to

$$(\forall x \in K) \qquad \inf_{\alpha\in\Delta} \inf_{a\in\mathcal{G}_f} p_{\alpha,a}(x) \geq 0.$$

To simplify this expression consider any $x \in K$, then

$$\begin{aligned}
\inf_{\alpha\in\Delta} \inf_{a\in\mathcal{G}_f} p_{\alpha,a}(x) &= \inf_{\alpha\in\Delta} \inf_{a\in\mathcal{G}_f} (p_\alpha(x) + \sup_{\lambda\in\Lambda} p_{a(\lambda)}(x)) \\
&= \inf_{a\in\mathcal{G}_f} p_\alpha(x) + \inf_{a\in\mathcal{G}_f} \sup_{\lambda\in\Lambda} p_{a(\lambda)}(x) \\
&= f(x) + \inf_{a\in\mathcal{G}_f} \sup_{\lambda\in\Lambda} p_{a(\lambda)}(x).
\end{aligned}$$

We now apply Lemma 3.1 with $t_\lambda(\alpha_\lambda) = p_{\alpha_\lambda}(x)$ (where $\alpha_\lambda = a(\lambda)$). Thus,

$$\inf_{a\in\mathcal{G}_f} \sup_{\lambda\in\Lambda} p_{a(\lambda)}(x) = \sup_{\lambda\in\Lambda} \inf_{a\in\mathcal{G}_f} p_{a(\lambda)}(x) = \sup_{\lambda\in\Lambda} \lambda g(x).$$

Hence,

$$\inf_{\alpha\in\Delta} \inf_{a\in\mathcal{G}_f} p_{\alpha,a}(x) = f(x) + \sup_{\lambda\in\Lambda} \lambda g(x),$$

so that condition (ii) is equivalent to the following:

$$(\forall x \in K) \qquad f(x) + \sup_{\lambda\in\Lambda} \lambda g(x) \geq 0. \tag{3}$$

We now verify that (3) implies (i) for every subset $\Lambda \in S^*$. If $g(x) \in -S$ then $\lambda g(x) \leq 0$ for all $\lambda \in \Lambda$ and so $\sup_{\lambda\in\Lambda} \lambda g(x) \leq 0$. Therefore, by (3), $f(x) \geq 0$ as required.

Now assume that cl co $\Lambda = S^*$, then

$$\sup_{\lambda\in\Lambda} \lambda g(x) = \sup_{\lambda\in S^*} \lambda g(x) = \begin{cases} 0 & \text{if } g(x) \in -S \\ +\infty & \text{if } g(x) \notin -S. \end{cases}$$

If (3) is not true then there is an $x \in K$ such that $f(x) + \sup_{\lambda\in\Lambda} \lambda g(x) < 0$. Thus we have $\sup_{\lambda\in\Lambda} \lambda g(x) < +\infty$. Hence $g(x) \in -S$ and $\sup_{\lambda\in\Lambda} \lambda g(x) = 0$. So it follows that $f(x) < 0$ and (i) is not true. ∎

COROLLARY 3.1. *Suppose that $K = X$ and f is* inf-$\mathrm{SL}_C(X)$ *with g as in Theorem* 3.1. *Then the following are equivalent:*

(i) $g(x) \in -S \Rightarrow f(x) \geq 0$.

(ii) For each $\alpha \in \Delta$ *and each selection* $(\alpha_\lambda) \in \prod_{\lambda\in\Lambda} \Delta(\lambda)$ *we have:*

$$0 \in \underline{\partial} p_\alpha + \operatorname{cl}\operatorname{co} \bigcup_{\lambda\in S^*} \underline{\partial} p_{\alpha_\lambda}.$$

Ishizuka [11] has produced a related finite-dimensional Farkas result for finite systems of functions possessing exhaustive families of lower concave approximations rather than upper convex approximations as in Corollary 3.1.

Suppose now that f is defined on a closed convex and bounded subset $X_0 \subseteq X$ and $g: X_0 \to Y$. Assume that f is inf-convex on X_0, so that, for all $x \in X_0$, $f(x) = \inf_{\alpha\in\Delta} h_\alpha(x)$ where $h_\alpha \in \Gamma(X_0)$. Furthermore, assume that there is a set $\Lambda \subseteq S^*$ such that the function $\lambda g(x)$ is inf-convex on X_0 for each $\lambda \in \Lambda$. Thus $\lambda g(x) = \inf_{\alpha_\lambda\in\Delta(\lambda)} h_{\alpha_\lambda}(x)$, with $h_{\alpha_\lambda} \in \Gamma(X_0)$. We also assume throughout that, for all α, α_λ, $\operatorname{dom} h_\alpha = \operatorname{dom} h_{\alpha_\lambda} = X_0$.

Let $K_0 = \{(\mu x, \mu): x \in X_0, \mu > 0\}$ and $K = K_0 \cup \{(0, 0)\}$. Since X_0 is a bounded closed convex set it is easy to check that K is a closed convex cone. Define the following mappings, for $\alpha \in \Delta$, $\lambda \in \Lambda$, $\alpha_\lambda \in \Delta(\lambda)$:

$$\begin{aligned}
G(x, \mu) &= \mu g(x/\mu), && (x, \mu) \in K_0,\ G(0, 0) = 0.\\
F(x, \mu) &= \mu f(x/\mu), && (x, \mu) \in K_0,\ F(0, 0) = 0.\\
H_\alpha(x, \mu) &= \mu h_\alpha(x/\mu), && (x, \mu) \in K_0,\ H_\alpha(0, 0) = 0.\\
H_{\alpha_\lambda}(x, \mu) &= \mu h_{\alpha_\lambda}(x/\mu), && (x, \mu) \in K_0,\ H_{\alpha_\lambda}(0, 0) = 0.
\end{aligned}$$

Also, $H_\alpha(x, \mu) = H_{\alpha_\lambda}(x, \mu) = +\infty$ if $(x, \mu) \notin K$. It is clear that

$$\liminf_{(x,\mu)\to(0,0)} H_\alpha(x, \mu) = \liminf_{(x,\mu)\to(0,0)} H_{\alpha_\lambda}(x, \mu) = 0.$$

Since K is closed the functions H_α and H_{α_λ} are l.s.c. Consider the following statement.

For each $\alpha \in \Delta$ and each selection $(\alpha_\lambda) \in \prod_{\lambda\in\Lambda} \Delta(\lambda)$

$$0 \in \operatorname{cl}\left(\underline{\partial} H_\alpha + \operatorname{co} \bigcup_{\lambda\in\Lambda} \underline{\partial} H_{\alpha_\lambda} - K^*\right). \tag{4}$$

Since $\underset{\sim}{\partial} H_\alpha = \mathscr{A}h_\alpha$, $\underset{\sim}{\partial} H_{\alpha_\lambda} = \mathscr{A}h_{\alpha_\lambda}$, we can rewrite (4) as follows:

$$0 \in \operatorname{cl}\left(\mathscr{A}h_\alpha + \operatorname{co}\bigcup_{\lambda\in\Lambda}\mathscr{A}h_{\alpha_\lambda} - K^*\right) \tag{5}$$

We now fix $\alpha \in \Delta$ and $\alpha_\lambda \in \Delta(\lambda)$. For this selection the inclusion (4) is equivalent (following the proof of Theorem 3.1) to

$$(\forall(x,\mu)\in K) \qquad H_\alpha(x,\mu) + \sup_{\lambda\in\Lambda} H_{\alpha_\lambda}(x,\mu) \geq 0 \tag{6}$$

or equivalently,

$$(\forall x\in X_0) \qquad h_\alpha(x) + \sup_{\lambda\in\Lambda} h_{\alpha_\lambda}(x) \geq 0. \tag{7}$$

The inequality in (7) can be written, equivalently, as follows:

$$(\forall\varepsilon > 0)(\forall x\in X_0)(\exists\lambda_0\in\Lambda) \qquad h_\alpha(x) + h_{\alpha_{\lambda_0}}(x) \geq -\varepsilon. \tag{8}$$

Following the argument in the proof of Theorem 3.1 we have that statement (4) can be written in the form

$$(\forall(x,\mu)\in K) \qquad F(x,\mu) + \sup_{\lambda\in\Lambda}\lambda G(x,\mu) \geq 0 \tag{9}$$

or equivalently,

$$(\forall x\in X_0) \qquad f(x) + \sup_{\lambda\in\Lambda}\lambda g(x) \geq 0. \tag{10}$$

Now consider the statement

$$(x,\mu)\in K, \qquad G(x,\mu)\in -S \Rightarrow F(x,\mu)\geq 0. \tag{11}$$

If $(x, \mu) \in K$ then either $(x, \mu) \in K_0$ or $(x, \mu) = (0, 0)$. Since $G(0, 0) = 0$ and $F(0, 0) = 0$ (by definition) we can say that (11) is equivalent to

$$(x,\mu)\in K_0, \qquad G(x,\mu)\in -S \Rightarrow F(x,\mu)\geq 0 \tag{12}$$

or, equivalently,

$$x\in X_0, \qquad g(x)\in -S \Rightarrow f(x)\geq 0. \tag{13}$$

Theorem 3.1 now shows that (9) is equivalent to (11). This argument yields the following extension of Theorem 3.1.

THEOREM 3.2. *Let X_0, Λ, f, g, h_α, and h_{α_λ} be as above. Consider the statements:*

(i) $x \in X_0$, $g(x) \in -S \Rightarrow f(x) \geq 0$.

(ii) *For all $\alpha \in \Delta$ and each selection $(\alpha_\lambda) \in \Pi_{\lambda \in \Lambda} \Delta(\lambda)$ we have*

$$0 \in \operatorname{cl}\left(\mathscr{A}h_\alpha + \operatorname{co} \bigcup_{\lambda \in \Lambda} \mathscr{A}h_{\alpha_\lambda} - K^*\right)$$

where $K = \{(\mu x, \mu) : x \in X_0, \mu \geq 0\}$.

(iii) *For all $\alpha \in \Delta$ and each selection $(\alpha_\lambda) \in \Pi_{\lambda \in \Lambda} \Delta(\lambda)$ we have*

$$(\forall x \in X_0) \qquad h_\alpha(x) + \sup_{\lambda \in \Lambda} h_{\alpha_\lambda}(x) \geq 0.$$

Then (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i). *If the set Λ is such that* $\operatorname{cl} \operatorname{co} \Lambda = S^*$ *then* (i) $\Rightarrow$ (ii).

The formula involved in (ii) above involves the cone K^*. If $0 \in \operatorname{int} X_0$ then

$$\begin{aligned} K^* &= \{(v, c) \in X' \times \mathbb{R} : (\forall x \in K_0)\, v(x) + c \geq 0\} \\ &= \{(v, c) : (\forall x \in X_0)\, v(x) + c \geq 0\}. \end{aligned}$$

Since $0 \in \operatorname{int} X_0$ there is a neighbourhood V of zero such that $v(x) + c \geq 0$ for all $x \in V$. Thus, since $\inf_{x \in V} v(x) < 0$, it follows that $c > 0$. Therefore,

$$(v, c) \in K^* \Leftrightarrow (\forall x \in X_0) - v(x) \leq c \Leftrightarrow (\forall x \in X_0) - \frac{v(x)}{c} \leq 1 \Leftrightarrow -\frac{v}{c} \in (X_0)^0,$$

where $(X_0)^0$ denotes the polar set of X_0. Thus

$$-K^* = \{(v, c) : \frac{v}{c} \in (X_0)^0, c > 0\}.$$

Consequently, (ii) in Theorem 3.2 can be written as follows:

$$0 \in \operatorname{cl}\left(\mathscr{A}h_\alpha + \operatorname{co} \bigcup_{\lambda \in \Lambda} \mathscr{A}h_{\alpha_\lambda} + \left\{(v, c) : \frac{v}{c} \in (X_0)^0, c > 0\right\}\right).$$

4. SOLVABILITY THEOREMS WITH A REGULARITY CONDITION

We now turn our attention to solvability theorems under a general regularity condition. In particular, we will show that the solvability results in [16] and [15] for systems of convex-like functions have analogues for systems of min-convex functions under similar regularity assumptions.

Recall that a function f is *convex-like* on a set X_0 if

$$(\exists \alpha \in (0,1))(\forall x_1, x_2 \in X_0)(\exists x_3 \in X_0) \qquad f(x_3) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

See Jeyakumar and Gwinner [15], Paeck [23], and Pomerol [24] for related definitions and properties.

To show that the results are independent the following example provides a system of min-convex functions which is not convex-like.

EXAMPLE 4.1. Let $f: [-1, 1] \to \mathbb{R}$, $g: [-1, 1] \to \mathbb{R}$ where $f(x) = \min \{0, x\}$ and $g(x) = \min \{0, -x\}$. Clearly f and g are min-convex. Define the set

$$\Omega = \{(a, b) \in \mathbb{R}^2 : (\exists x \in [-1, 1]) f(x) \leq a, g(x) \leq b\}.$$

As noted in [16], if (f, g) is convex-like then cl Ω is convex (that is, Ω is *nearly convex*). It is straightforward to show that $(-1, 0) \in \Omega$ and $(0, -1) \in \Omega$, however $(-\alpha, -(1-\alpha)) \notin \Omega$ for each $\alpha \in (0, 1)$. Thus Ω is not convex and, since it is closed, it is not nearly convex [16]. Hence (f, g) is not convex-like.

Let I be a possibly infinite index set. Let $g: X_0 \times I \to \mathbb{R}$ and $f_0: X_0 \to \mathbb{R}$ be functions with f_0 and, for each $i \in I$, $g(\cdot, i)$ min-convex. Thus there are nonempty index sets Δ_0 and, for each $i \in I$, $\Delta(i)$ such that, for all $x \in X_0$,

$$f_0(x) = \min_{\alpha \in \Delta_0} f_\alpha(x), \qquad g(x, i) = \min_{\alpha_i \in \Delta(i)} g_{\alpha_i}(x),$$

where f_α and g_{α_i} are convex functions (for each α and α_i).

We are interested in the following system:

$$f_0(x) < 0, g(x, i) \leq 0, \qquad i \in I, x \in X_0. \tag{14}$$

We let $\mathbb{R}^I := \prod_I \mathbb{R}$ denote the product space with the usual product topology. This is a locally convex Hausdorff topological vector space (see [18]). The topological dual space of $\mathbb{R}^I$ is the generalized finite sequence space of all functions $w: I \to \mathbb{R}$ with finite support. The set $\mathbb{R}^I_+$ denotes the convex cone of nonnegative functions on I. The corresponding dual cone of $\mathbb{R}^I_+$ is the set

$$\Lambda = \{\lambda = (\lambda_i)_{i \in I} : (\exists \text{ finite set } J \subseteq I)(\forall i \in I \backslash J)\ \lambda_i = 0, (\forall i \in J)\ \lambda_i \geq 0\}.$$

In the following we will require the following regularity condition, first introduced in [15, 16] for systems of convex-like functions.

DEFINITION 4.1. The system (14) is said to satisfy the *closure regularity condition* if there is a neighbourhood U of zero in $\mathbb{R}^l$ and a constant $\gamma > 0$ such that

$$\Omega_0 \cap \overline{U} \times (-\infty, \gamma]$$

is a nonempty closed set in $\mathbb{R}^l \times \mathbb{R}$, where

$$\Omega_0 = \{(u, r) \in \mathbb{R}^l \times \mathbb{R} : (\exists x \in X_0) f_0(x) \leq r, (\forall i \in I)\, g(x, i) \leq u_i\}.$$

THEOREM 4.1. *Let the system (14) satisfy the closure regularity condition. Then exactly one of the following systems is satisfied:*

$$(\exists x \in X_0) \qquad f_0(x) < 0, (\forall i \in I)\, g(x, i) \leq 0, \tag{15}$$

$$(\forall \alpha \in \Delta_0)(\forall (\alpha_i) \in \prod_{i \in I} \Delta(i))(\forall \varepsilon > 0)(\exists (\lambda, \tau) \in \Lambda \times \mathbb{R}_+, (\lambda, \tau) \neq (0, 0))(\forall x \in X_0)$$

$$\tau(f_\alpha(x) + \varepsilon) + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > 0. \tag{16}$$

Proof. Suppose both (15) and (16) are solvable. Then there is an $x_0 \in X_0$ such that

$$f_0(x_0) < 0, \qquad (\forall i \in I)\, g(x_0, i) \leq 0.$$

We now select $\bar{\alpha} \in \Delta_0$, $\bar{\alpha}_i \in \Delta(i)$, for each $i \in I$, such that $f_{\bar{\alpha}}(x_0) = f_0(x_0)$ and $g_{\bar{\alpha}_i}(x_0) = g(x_0, i)$. Now for this selection and $\varepsilon = -f_0(x_0) = -f_{\bar{\alpha}}(x_0)$ we have, by (16), the existence of $\lambda \in \Lambda$ such that

$$\sum_{i \in I} \lambda_i g_{\bar{\alpha}_i}(x_0) > 0.$$

This is a contradiction since $\lambda_i \geq 0$ and $g_{\bar{\alpha}_i}(x_0) \leq 0$ by construction. Thus both systems cannot be solvable.

Now suppose that (15) is not solvable. Then for any selection $(\alpha, (\alpha_i))$ the following system is not solvable:

$$f_\alpha(x) < 0, \qquad (\forall i \in I)\, g_{\alpha_i}(x) \leq 0. \tag{17}$$

As in [16] it follows, by the closure regularity condition, that for any $\varepsilon > 0$,

$$(0, -\varepsilon) \notin \operatorname{cl} \Omega_0. \tag{18}$$

Otherwise, there is a net $(u_\kappa, r_\kappa) \in \Omega_0$ such that $u_\kappa \to 0$ and $r_\kappa \to -\varepsilon$. Since U is a neighbourhood of zero and $\gamma > 0$ we can select a subnet

$(u_\delta, r_\delta) \in \Omega_0 \cap \overline{U} \times (-\infty, \gamma]$. By the closure regularity condition it follows that

$$(0, -\varepsilon) = \lim_{\delta} (u_\delta, r_\delta) \in \Omega_0 \cap \overline{U} \times (-\infty, \gamma].$$

Thus there is a $x_0 \in X_0$ such that $f(x_0) \leq -\varepsilon < 0$ and, for all i, $g(x_0, i) \leq 0$, a contradiction.

Now we define for the system (17) the set

$$\Omega_s = \{(u, r) \in \mathbb{R}^I \times \mathbb{R} : (\exists x \in X_0) f_\alpha(x) \leq r, (\forall i \in I)\, g_{\alpha_i}(x) \leq u_i\}.$$

By the min-convexity assumptions it follows that $\Omega_s \subseteq \Omega_0$. Since (17) is a convex system the set Ω_s is convex and by (18) it follows that, for any $\varepsilon > 0$, $(0, -\varepsilon) \notin \operatorname{cl} \Omega_s$. By the separation theorem there exists nonzero $(\lambda, \tau) \in (\mathbb{R}^I)' \times \mathbb{R}$ such that, for each $(u, r) \in \Omega_0$, $\tau r + \lambda(u) > -\tau\varepsilon$. Thus,

$$(\forall x \in X_0)\, \tau(f_\alpha(x)) + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > -\tau\varepsilon.$$

It can be shown, as in [16], that $\lambda \in (\mathbb{R}^I_+)^* = \Lambda$ and $\tau \geq 0$. Hence (16) is solvable as required. Since the selection was arbitrary the result follows. ∎

Remark 4.1. Various sufficient conditions for the closure regularity condition to be satisfied have been discussed in [16]; for example, it is valid if X_0 is compact and each of the functions f_0 and $g(\cdot, i)$ are lower semicontinuous.

In Theorem 4.1 the min-convexity condition can be weakened to any condition which will guarantee that the set Ω_s is convex (to allow a separation argument to be applied). In addition f_0 need only be inf-convex for Theorem 4.1 to remain valid. We now present an extension of Farkas' lemma to systems of min-convex functions.

THEOREM 4.2. *Assume the same hypotheses as in Theorem* 4.1 *for system* (14). *In addition, let the following be satisfied:*

$$(\exists x_0 \in X_0)(\forall i \in I) \qquad \sup_{\alpha_i \in \Delta(i)} g_{\alpha_i}(x_0) \leq 0. \tag{19}$$

Then the following statements are equivalent:

(i) $(\forall i \in I)\; g(x, i) \leq 0 \Rightarrow f_0(x) \geq 0$.

(ii) $(\forall \alpha \in \Delta_0)(\forall (\alpha_i) \in \prod_{i \in I} \Delta(i))(\forall \varepsilon > 0)(\exists \lambda \in \Lambda)(\forall x \in X_0)$

$$f_\alpha(x) + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > -\varepsilon. \tag{20}$$

Proof. Clearly (i) is equivalent to (15) not being solvable which, by Theorem 4.1, is equivalent to solvability of (16). Thus for any selection $(\alpha, (\alpha_i))$ and any $\varepsilon > 0$ there exist nonzero $(\lambda, \tau) \in \Lambda \times \mathbb{R}_+$ such that

$$\tau(f_\alpha(x) + \varepsilon) + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > 0. \tag{21}$$

Now suppose that $\tau = 0$, then we obtain a contradiction to (19), since $\lambda_i \geq 0$, by substituting $x = x_0$. Thus $\tau \neq 0$ can be assumed and we can take $\tau = 1$ for convenience. ▌

Remark 4.2. Note that Theorem 4.2 above should be compared with Theorem 3.2. In particular, note that Theorem 4.2(ii) shows that, for all $x \in X_0$,

$$f_\alpha(x) + \sup_{\lambda \in \Lambda} \sum_{i \in I} \lambda_i g_{\alpha_i}(x) \geq 0.$$

Essentially, the regularity condition has enabled certain quantifiers to be interchanged in comparing Theorem 3.2(iii) and Theorem 4.2(ii).

The following generalized Gordan alternative theorem easily follows from Theorem 4.1.

THEOREM 4.3. *Let I be an arbitrary index set with $g: X_0 \times I \to \mathbb{R}$ such that $g(\cdot, i)$ is min-convex for each $i \in I$. Suppose that there is a neighbourhood U of zero in $\mathbb{R}^I$ such that $\Omega \cap \overline{U}$ is nonempty and closed, where*

$$\Omega = \{u \in \mathbb{R}^I : (\exists x \in X_0)(\forall i \in I)\, g(x, i) \leq u_i\}.$$

Then exactly one of the following systems is solvable.

$$(\exists x \in X)(\forall i \in I)\, g(x, i) \leq 0. \tag{22}$$

$$(\forall (\alpha_i) \in \prod_{i \in I} \Delta(i))(\exists \lambda \in \Lambda_1)(\forall x \in X) \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > 0, \tag{23}$$

where $\Lambda_1 = \{\lambda \in \Lambda : \sum_{i \in I} \lambda_i = 1\}$.

Proof. Suppose both systems are solvable, then there is a $x_0 \in X_0$ such that $g(x_0, i) \leq 0$ for all $i \in I$. Now select $(\overline{\alpha}_i)$ with $g_{\overline{\alpha}_i}(x_0) = g(x_0, i)$. Then (23) is contradicted since $\lambda_i \geq 0$ and $g_{\overline{\alpha}_i}(x_0) \leq 0$, for all i.

Now if (22) is not solvable then (15) is not solvable with $f_0(x) := -\varepsilon_0 < 0$ (for all $x \in X$). Thus, (16), with $\varepsilon = \varepsilon_0 > 0$ and any selection (α_i), shows that there is a $\lambda \neq 0$ with

$$(\forall x \in X_0) \qquad \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > 0.$$

Dividing by $\Sigma_{i \in I} \lambda_i$ gives the desired result. ∎

We do not require the set X_0 to be convex or closed in the above results. We complete this discussion of solvability theorems under a regularity condition by considering two examples to illustrate the results. First we show that min-convex cannot be weakened, in general, to inf-convex in Theorem 4.3.

EXAMPLE 4.2. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sqrt{|x|}$. Then g is inf-convex but not min-convex. In this case it is easily seen that the regularity condition of Theorem 4.3 is valid however both (22) and (23) are satisfied in this case.

We now give a simple example which illustrates that Theorem 4.2 is applicable whereas the results of [16] are not.

EXAMPLE 4.3. Let g, $f_0 : \mathbb{R} \to \mathbb{R}$, $f_0(x) = \min\{0, x\}$, and $g(x) = \min\{0, -x\}$. As noted previously (f_0, g) is not convex-like. However, we can apply Theorem 4.2 to the system

$$f_0(x) < 0, \qquad g(x) \leq 0.$$

This system is solvable ($x = -1$, for example). We can show that the alternative system from Theorem 4.2 is not solvable by considering the possible selections given by the bounding functions $f_{0_1}(x) = 0$, $f_{0_2}(x) = x$, $g_1(x) = 0$, $g_2(x) = -x$. We are thus interested in the systems, for j, $k = 1, 2$, given by

$$(\forall x \in \mathbb{R}) \qquad f_{0_j}(x) + \gamma g_k(x) \geq 0, \gamma \geq 0.$$

Clearly this is not solvable with, for example, $j = 2$ and $k = 1$ and $x < 0$. Thus Theorem 4.2 is satisfied in this case. It is easily shown that the closure regularity condition is satisfied since $\Omega_0 = \{(a, 0) \in \mathbb{R}^2 : a \leq 0\} \cup \{(0, b) \in \mathbb{R}^2 : b \leq 0\}$.

5. APPLICATION TO LAGRANGIAN DUALITY

In this section we apply the solvability theorems from the previous section to infinite-dimensional programming problems involving min-convex functions. In particular, we investigate Lagrangian duality for a class of nonconvex programming problems using approximating convex subprograms. The regularity condition we will use is the closure regularity condition (see Definition 4.1).

Consider the programming problem:

$$\text{(P)} \qquad \mu = \inf\{f_0(x) : x \in C, (\forall i \in I)\, g(x, i) \le 0\},$$

where I is a possibly infinite index set and the functions f_0 and, for each $i \in I$, $g(\cdot, i)$ are min-convex. We will adopt the same terminology throughout this section as used in the preceding section, particularly with regard to selections and indexing sets. The *Lagrangian dual* to (P) is the problem:

$$\text{(D)} \qquad \nu = \sup\{\phi(\lambda) : \lambda \in \Lambda\},$$

where $\phi(\lambda) = \inf_{x \in C}(f_0(x) + \sum_{i \in I}\lambda_i g(x, i))$. It is well known that $\mu \ge \nu$, that is, weak duality holds between (P) and (D). Associated with (P) we have the following family of convex programs (one associated with each selection $s = (\alpha, (\alpha_i))$ where $\alpha \in \Delta_0$ and $(\alpha_i) \in \prod_{i \in I} \Delta(i))$ and their corresponding Lagrangian duals (we will use the subscript s to denote a subprogram dependent on some selection as indicated above). Thus

$$(\text{P}_s) \qquad \mu_s = \inf\{f_\alpha(x) : x \in C, (\forall i \in I)\, g_{\alpha_i}(x) \le 0\}.$$

$$(\text{D}_s) \qquad \nu_s = \sup\{\phi_s(\lambda) : \lambda \in \Lambda\},$$

where $\phi_s(\lambda) = \inf_{x \in C}(f_\alpha(x) + \sum_{i \in I}\lambda_i g_{\alpha_i}(x))$. As in the preceeding section we assume that the system (f_0, g) satisfies the closure regularity condition; that is, there is a neighbourhood U of zero in $\mathbb{R}^I$ and a $\gamma > 0$ such that $\Omega_0 \cap \overline{U} \times (-\infty, \gamma]$ is a nonempty closed set in $\mathbb{R}^I \times \mathbb{R}$, where

$$\Omega_0 = \{(u, r) \in \mathbb{R}^I \times \mathbb{R} : (\exists x \in C) f_0(x) \le r, (\forall i \in I)\, g(x, i) \le u_i\}.$$

For simplicity we assume that μ is finite (so that (P) is consistent) and, for each selection, μ_s is finite (so that each subprogram is consistent).

LEMMA 5.1. *Assume (f_0, g) satisfies the closure regularity condition and that*

$$(\exists x_0 \in C)(\forall i \in I) \sup_{\alpha_i \in \Delta(i)} g_{\alpha_i}(x_0) \le 0. \tag{24}$$

Then for each selection s we have

$$\nu \le \mu \le \nu_s \le \mu_s.$$

Proof. By weak duality $\nu \le \mu$ and $\nu_s \le \mu_s$. Take any selection $s = (\alpha, (\alpha_i))$, clearly $\mu \le \mu_s$ by min-convexity. Now, since μ is finite, there is no solution to the system

$$f_0(x) - \mu < 0 \qquad (\forall i \in I)\, g(x, i) \le 0, x \in C.$$

We can immediately apply, since (24) is valid, Theorem 4.2. Thus we have the following:

$$(\forall \alpha \in \Delta_0)(\forall (\alpha_i) \in \prod_{i \in I} \Delta(i))(\forall \varepsilon > 0)(\exists \lambda \in \Lambda)(\forall x \in C)$$

$$f_\alpha(x) - \mu + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > -\varepsilon.$$

Hence, taking any $\varepsilon > 0$, there is a $\lambda \in \Lambda$ such that, for all $x \in C$, $f_\alpha(x) + \sum_{i \in I} \lambda_i g_{\alpha_i}(x) > \mu - \varepsilon$. Taking the infimum over all $x \in C$ shows that $\phi_s(\lambda) \ge \mu - \varepsilon$. Hence, since $\varepsilon > 0$ was arbitrary, $\nu_s = \sup_{\lambda \in \Lambda} \phi_s(\lambda) \ge \mu$. Since weak duality holds between (P_s) and (D_s) the result follows. ▮

The additional condition (24) amounts to assuming that the feasible regions of the subprograms have a nonempty intersection. If $g(\cdot, i)$ is convex for each $i \in I$ then this condition is trivially satisfied by consistency of (P).

THEOREM 5.1. *Let the assumptions of Lemma 5.1 be satisfied. Then*

$$\nu \le \mu = \inf_s \nu_s = \inf_s \mu_s.$$

Proof. By Lemma 5.1 we need only show that $\mu = \inf_s \mu_s$. Since $\mu \le \mu_s$ for any selection s, $\mu \le \inf_s \mu_s$. Let $\varepsilon > 0$, then there is an $x_\varepsilon \in C$ with $g(x_\varepsilon, i) \le 0$ for all i and $\mu \le f_0(x_\varepsilon) < \mu + \varepsilon$. We now select $\bar{s} = (\bar{\alpha}, (\bar{\alpha}_i))$ such that $f_0(x_\varepsilon) = f_{\bar{\alpha}}(x_\varepsilon)$ and, for all i, $g(x_\varepsilon, i) = g_{\bar{\alpha}_i}(x_\varepsilon)$. Thus x_ε is feasible for $(\mathrm{P}_{\bar{s}})$ so that

$$\mu \le \mu_{\bar{s}} \le f_{\bar{\alpha}}(x_\varepsilon) = f_0(x_\varepsilon) < \mu + \varepsilon.$$

Hence the result follows since $\varepsilon > 0$ was arbitrary. ▮

We complete this section with a result concerning the *value function* to the program (P), namely, for $z \in \mathbb{R}^I$,

$$V(z) = \inf\{f_0(x) : x \in C, (\forall i \in I)\, g(x, i) \le z_i\}.$$

In the obvious way we denote the value function to the associated subprograms for each selection s by

$$V_s(z) = \inf\{f_\alpha(x) : x \in C, (\forall i \in I)\, g_{\alpha_i}(x) \leq z_i\}.$$

The following result follows in a fashion similar to Theorem 5.1.

THEOREM 5.2. *Let the assumptions of Lemma* 5.1 *be satisfied, then the following are valid:*

(i) *For all* $z \in \mathbb{R}^I$, $V(z) = \inf_s V_s(z)$.

(ii) $V(\cdot)$ *is inf-convex, since* V_s *is convex for each selection* s.

Consider the special case in which the index sets Δ, $\Delta(i)$, and I are finite. In this case the set of all selections is finite and consequently we can replace inf by min in Theorems 5.1 and 5.2. Furthermore, if we assume that $g(\cdot, i)$ is convex for each $i \in I$ then the problem (P) has been considered by Borwein [2, Example 5.2]. In this case Theorem 5.1 gives the following

$$\mu = \inf_{\alpha \in \Delta} \sup_{\lambda \in \Lambda} \phi_\alpha(\lambda),$$

where $f_0 = \inf_{\alpha \in \Delta} f_\alpha$ and $\phi_\alpha(\lambda) = \inf_{x \in C}\{f_\alpha(x) + \sum_{i \in I} \lambda_i g_i(x, i)\}$. To illustrate this result consider the following simple nonconvex example:

EXAMPLE 5.1. Let $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = \min\{x - 1, 0\}$, and $g(x) = -x$. We are interested in the primal problem:

$$\text{(P)} \qquad \mu = \min\{f(x) : g(x) \leq 0\}.$$

Clearly $\mu = -1$. It is not difficult to show that the value of the Lagrangian dual problem (D) in this case is $\nu = -\infty$. Thus, by the nonconvexity of f, we have a duality gap. However, as f is min-convex we can consider the two associated Lagrangian duals as follows:

$$(\mathrm{D}_i) \qquad \nu_i = \sup\{\phi_i(\lambda) : \lambda \geq 0\}, \qquad i = 1, 2,$$

where $\phi_1(\lambda) = \inf_{x \in \mathbb{R}}(x - 1 - \lambda x)$ and $\phi_2(\lambda) = \inf_{x \in \mathbb{R}}(-\lambda x)$. It is straightforward to show that $\nu_1 = -1$ and $\nu_2 = 0$, thus

$$-\infty = \nu < -1 = \mu = \min_i \nu_i.$$

The system (f, g) clearly satisfies all the assumptions of Lemma 5.1.

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